2017

AP® CollegeBoard

AP Calculus BC

Sample Student Responses and Scoring Commentary

Inside:

- ☑ Free Response Question 1
- ☑ Scoring Guideline
- ☑ Student Samples
- ☑ Scoring Commentary

Question 1

	1 : units in parts (a), (c), and (d)
(a) Volume $= \int_0^{10} A(h)\,dh$ $\approx (2-0)\cdot A(0) + (5-2)\cdot A(2) + (10-5)\cdot A(5)$ $= 2\cdot 50.3 + 3\cdot 14.4 + 5\cdot 6.5$ $= 176.3$ cubic feet	$2 : \begin{cases} 1 : \text{left Riemann sum} \\ 1 : \text{approximation} \end{cases}$
(b) The approximation in part (a) is an overestimate because a left Riemann sum is used and A is decreasing.	1 : overestimate with reason
(c) $\int_0^{10} f(h)\,dh = 101.325338$ The volume is 101.325 cubic feet.	$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$
(d) Using the model, $V(h) = \int_0^h f(x)\,dx$. $\left.\frac{dV}{dt}\right\|_{h=5} = \left[\frac{dV}{dh}\cdot\frac{dh}{dt}\right]_{h=5}$ $= \left[f(h)\cdot\frac{dh}{dt}\right]_{h=5}$ $= f(5)\cdot 0.26 = 1.694419$ When $h = 5$, the volume of water is changing at a rate of 1.694 cubic feet per minute.	$3 : \begin{cases} 2 : \frac{dV}{dt} \\ 1 : \text{answer} \end{cases}$

h (feet)	0	2	5	10
$A(h)$ (square feet)	50.3	14.4	6.5	2.9

1. A tank has a height of 10 feet. The area of the horizontal cross section of the tank at height h feet is given by the function A, where $A(h)$ is measured in square feet. The function A is continuous and decreases as h increases. Selected values for $A(h)$ are given in the table above.

(a) Use a left Riemann sum with the three subintervals indicated by the data in the table to approximate the volume of the tank. Indicate units of measure.

$V \approx [2(50.3) + 3(14.4) + 5(6.5)]$

$\approx 176.3 \text{ ft}^3$

(b) Does the approximation in part (a) overestimate or underestimate the volume of the tank? Explain your reasoning.

The approximation in part a is an overestimate because A is a decreasing function.

Continue question 1 on page 5.

(c) The area, in square feet, of the horizontal cross section at height h feet is modeled by the function f given by $f(h) = \dfrac{50.3}{e^{0.2h} + h}$. Based on this model, find the volume of the tank. Indicate units of measure.

$$V = \int_0^{10} \left(\frac{50.3}{e^{0.2h} + h} \right) dh$$

$$\approx 101.325 \text{ ft}^3$$

(d) Water is pumped into the tank. When the height of the water is 5 feet, the height is increasing at the rate of 0.26 foot per minute. Using the model from part (c), find the rate at which the volume of water is changing with respect to time when the height of the water is 5 feet. Indicate units of measure.

$h = 5$ ft

$\frac{dh}{dt} = 0.26$ ft/min

$\frac{dV}{dt} = ?$

$$V = \int_0^h \left(\frac{50.3}{e^{0.2h} + h} \right) dh$$

$$\frac{dV}{dt} = \frac{50.3}{e^{0.2h} + h} \cdot \frac{dh}{dt}$$

$$= \frac{50.3}{e^{0.2(5)} + (5)} \cdot 0.26$$

$$\approx 1.694 \text{ ft}^3/\text{min}$$

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h (feet)	0	2	5	10
$A(h)$ (square feet)	50.3	14.4	6.5	2.9

1. A tank has a height of 10 feet. The area of the horizontal cross section of the tank at height h feet is given by the function A, where $A(h)$ is measured in square feet. The function A is continuous and decreases as h increases. Selected values for $A(h)$ are given in the table above.

(a) Use a left Riemann sum with the three subintervals indicated by the data in the table to approximate the volume of the tank. Indicate units of measure.

$$V = [A(0)](2) + [A(2)](3) + [A(5)](5)$$
$$= (50.3)(2) + (14.4)(3) + (6.5)(5) = 176.3 \text{ ft}^3$$

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(b) Does the approximation in part (a) overestimate or underestimate the volume of the tank? Explain your reasoning.

The volume is overestimated, because the A(h) is continuosly decreasing, so after the value of A(h) at the left end of each subinterval, A(h) is lower for the rest of the subinterval (compared to the left end). which means the actual volume should be lower than the approximation.

Continue question 1 on page 5.

(c) The area, in square feet, of the horizontal cross section at height h feet is modeled by the function f given by $f(h) = \frac{50.3}{e^{0.2h} + h}$. Based on this model, find the volume of the tank. Indicate units of measure.

Volume equals to $\int_0^{10} f(h)\,dh \doteq 101 \text{ ft}^3$

(use of graphing calculator)

(d) Water is pumped into the tank. When the height of the water is 5 feet, the height is increasing at the rate of 0.26 foot per minute. Using the model from part (c), find the rate at which the volume of water is changing with respect to time when the height of the water is 5 feet. Indicate units of measure.

$V = \int_0^5 f(h)\,dh$, $\frac{dV}{dt} = [f(5) - f(0)]\frac{dh}{dt}$

$\therefore \frac{dV}{dt} = (6.52 - 50.3)(0.26) = 11.3828 \text{ ft}^3/\text{minute}$

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h (feet)	0	2	5	10
$A(h)$ (square feet)	50.3	14.4	6.5	2.9

1. A tank has a height of 10 feet. The area of the horizontal cross section of the tank at height h feet is given by the function A, where $A(h)$ is measured in square feet. The function A is continuous and decreases as h increases. Selected values for $A(h)$ are given in the table above.

(a) Use a left Riemann sum with the three subintervals indicated by the data in the table to approximate the volume of the tank. Indicate units of measure.

$2(14.4)+3(6.5)+5(2.9)$

$28.8 + 19.5 + 14.5 = 62.8\ ft^3$

The volume of the tank is the $\int_0^{10} A(h)$, therefore using a left Riemann sum, the volume of the tank is $\boxed{62.8\ ft^3}$.

(b) Does the approximation in part (a) overestimate or underestimate the volume of the tank? Explain your reasoning.

Because the function A is decreasing over the interval between $0 \le h \le 10$, the LRAM is an under-estimate of the volume in the tank.

Continue question 1 on page 5.

(c) The area, in square feet, of the horizontal cross section at height h feet is modeled by the function f given by $f(h) = \frac{50.3}{e^{0.2h} + h}$. Based on this model, find the volume of the tank. Indicate units of measure.

The Volume of the tank is the $\int f(h)\,dh$.

$$\int_0^{10} \frac{50.3}{e^{.2h}+h}\,dh = 101.325 \text{ ft}^3$$

(d) Water is pumped into the tank. When the height of the water is 5 feet, the height is increasing at the rate of 0.26 foot per minute. Using the model from part (c), find the rate at which the volume of water is changing with respect to time when the height of the water is 5 feet. Indicate units of measure.

$h=5$

$\frac{dh}{dt} = .26$ ft/min

$\frac{dv}{dt} = f(h) = \frac{50.3}{e^{.2h}+h}$ when $h=5 = 6.517$ ft/min²

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Question 1

Overview

In this problem students were presented with a tank that has a height of 10 feet. The area of the horizontal cross section of the tank at height h feet is given by a continuous and decreasing function A, where $A(h)$ is measured in square feet. Values of $A(h)$ for heights $h = 0,\ 2, 5,$ and 10 are supplied in a table. In part (a) students were asked to approximate the volume of the tank using a left Riemann sum and indicate the units of measure. Students needed to respond by incorporating data from the table in a left Riemann sum expression approximating $\int_0^{10} A(h)\,dh$ using the subintervals $[0, 2],\ [2, 5],$ and $[5, 10].$ [LO 3.2B/EK 3.2B2] In part (b) students needed to explain that a left Riemann sum approximation for the definite integral of a continuous, decreasing function overestimates the value of the integral. [LO 3.2B/EK 3.2B2] In part (c) the function f given by $f(h) = \dfrac{50.3}{e^{0.2h} + h}$ is presented as a model for the area, in square feet, of the horizontal cross section at height h feet. Students were asked to find the volume of the tank using this model, again indicating units of measure. Using the model f for cross-sectional areas of the tank, students needed to express the volume of the tank as $\int_0^{10} f(h)\,dh$ and use the graphing calculator to produce a numeric value for this integral. [LO 3.4D/EK 3.4D2] In part (d) water is pumped into the tank so that the water's height is increasing at the rate of 0.26 foot per minute at the instant when the height of the water is 5 feet. Students were asked to use the model from part (c) to find the rate at which the volume of water is changing with respect to time when the height of the water is 5 feet, again indicating units of measure. Students needed to realize that the volume of water in the tank, as a function of its height h, is given by $V(h) = \int_0^h f(x)\,dx$ and then use the Fundamental Theorem of Calculus to find that the rate of change of the volume of water with respect to its height is given by $V'(h) = f(h).$ Then, using the chain rule for derivatives, students needed to relate the rates of change of volume with respect to time and height and the rate of change of the water's height with respect to time. Information in the problem suffices to be able to find these rates when the water's height is 5 feet. [LO 2.3C/EK 2.3C2, LO 3.3A/EK 3.3A2] This problem incorporates the following Mathematical Practices for AP Calculus (MPACs): reasoning with definitions and theorems, connecting concepts, implementing algebraic/computational processes, connecting multiple representations, building notational fluency, and communicating.

Sample: 1A
Score: 9

The response earned all 9 points: the units point, 2 points in part (a), 1 point in part (b), 2 points in part (c), and 3 points in part (d). The units point was earned because the units of ft^3 in parts (a) and (c) and ft^3/min in part (d) are all correct. In part (a) the left Riemann sum point was earned by the numerical expression in the first line. This expression would have also earned the approximation point without simplification. The student chooses to simplify, does so correctly, and thus earned the approximation point. In part (b) the statement "overestimate because A is a decreasing function" earned the point. In part (c) the definite integral earned the integral point, and 101.325 earned the answer point. In part (d) the second line on the right earned the 2 points for $\dfrac{dV}{dt}$. The third line on the right would have earned the answer point without simplification. The student chooses to give a final answer of 1.694 that is computed correctly and earned the answer point.

Question 1 (continued)

Sample: 1B
Score: 6

The response earned 6 points: the units point, 2 points in part (a), 1 point in part (b), 1 point in part (c), and 1 point in part (d). The units point was earned because the units of ft^3 in parts (a) and (c) and ft^3/minute in part (d) are all correct. In part (a) the left Riemann sum point was earned by the symbolic expression in the first line. The approximation point was earned by the second line. The numerical expression in the second line would have earned the approximation point without simplification to 176.3. The student chooses to simplify, does so correctly, and thus earned the approximation point. In part (b) the response of "overestimated" with the reason that "$A(h)$ is continuously decreasing" earned the point. In part (c) the expression $\int_0^{10} f(h)\,dh$ earned the integral point. The answer of 101 did not earn the answer point because the result is not accurate to three places after the decimal point. In part (d) the equation $\frac{dV}{dt} = [f(5) - f(0)]\frac{dh}{dt}$ earned 1 of the 2 $\frac{dV}{dt}$ points for using the chain rule. The student has made an error in the application of the Fundamental Theorem of Calculus. The numerical evaluation of the student's expression for $\frac{dV}{dt}$ is not eligible to earn the answer point.

Sample: 1C
Score: 3

The response earned 3 points: no units point, no points in part (a), 1 point in part (b), 2 points in part (c), and no points in part (d). The units of ft/min^2 in part (d) are incorrect, so the student did not earn the units point. Because the student uses a right Riemann sum, no points were earned in part (a). The confusion of left with right has implications in part (b). Because the setup in part (a) for a right Riemann sum is accurate, the student is eligible to earn the point in part (b) if the response is consistent for a right Riemann sum. In part (b) the implication that A is decreasing leads to an "under-estimate" is consistent with a right Riemann sum. Thus, the student earned the point. In part (c) $\int_0^{10} \frac{50.3}{e^{.2h} + h}\,dh$ earned the integral point, and 101.325 earned the answer point. In part (d) no points were earned because the statement of $\frac{dV}{dt} = f(h)$ is incorrect, and the remaining work only implies, incorrectly, that $\frac{dV}{dt} = f(5) = 6.517.$ The answer point was not earned because the student's expression for $\frac{dV}{dt}$ is not eligible to earn the answer point.

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Inside:

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Question 2

(a) $\frac{1}{2}\int_0^{\pi/2} (f(\theta))^2 \, d\theta = 0.648414$

The area of R is 0.648.

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) $\int_0^k \left((g(\theta))^2 - (f(\theta))^2 \right) d\theta = \frac{1}{2}\int_0^{\pi/2} \left((g(\theta))^2 - (f(\theta))^2 \right) d\theta$

— OR —

$\int_0^k \left((g(\theta))^2 - (f(\theta))^2 \right) d\theta = \int_k^{\pi/2} \left((g(\theta))^2 - (f(\theta))^2 \right) d\theta$

$2 : \begin{cases} 1 : \text{integral expression} \\ \quad \text{for one region} \\ 1 : \text{equation} \end{cases}$

(c) $w(\theta) = g(\theta) - f(\theta)$

$$w_A = \frac{\int_0^{\pi/2} w(\theta) \, d\theta}{\frac{\pi}{2} - 0} = 0.485446$$

The average value of $w(\theta)$ on the interval $\left[0, \frac{\pi}{2}\right]$ is 0.485.

$3 : \begin{cases} 1 : w(\theta) \\ 1 : \text{integral} \\ 1 : \text{average value} \end{cases}$

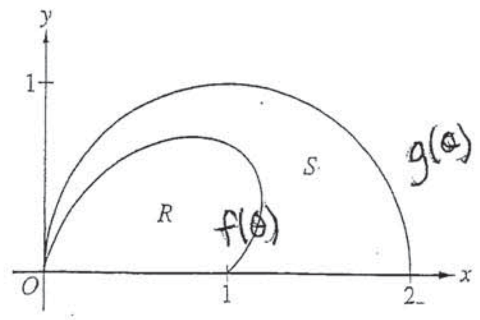
(d) $w(\theta) = w_A$ for $0 \le \theta \le \frac{\pi}{2} \Rightarrow \theta = 0.517688$

$w(\theta) = w_A$ at $\theta = 0.518$ (or 0.517).

$w'(0.518) < 0 \Rightarrow w(\theta)$ is decreasing at $\theta = 0.518$.

$2 : \begin{cases} 1 : \text{solves } w(\theta) = w_A \\ 1 : \text{answer with reason} \end{cases}$

2A,



2. The figure above shows the polar curves $r = f(\theta) = 1 + \sin\theta\cos(2\theta)$ and $r = g(\theta) = 2\cos\theta$ for $0 \le \theta \le \frac{\pi}{2}$. Let R be the region in the first quadrant bounded by the curve $r = f(\theta)$ and the x-axis. Let S be the region in the first quadrant bounded by the curve $r = f(\theta)$, the curve $r = g(\theta)$, and the x-axis.

(a) Find the area of R.

$$R = \frac{1}{2}\int_0^{\frac{\pi}{2}} f(\theta)^2\,d\theta \approx 0.648$$

(b) The ray $\theta = k$, where $0 < k < \frac{\pi}{2}$, divides S into two regions of equal area. Write, but do not solve, an equation involving one or more integrals whose solution gives the value of k.

$$\frac{1}{2}\int_0^{k}\left[g(\theta)^2 - f(\theta)^2\right]d\theta - \frac{1}{2}\int_k^{\frac{\pi}{2}}\left[g(\theta)^2 - f(\theta)^2\right]d\theta = 0$$

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Continue question 2 on page 7.

2A2

(c) For each θ, $0 \le \theta \le \frac{\pi}{2}$, let $w(\theta)$ be the distance between the points with polar coordinates $(f(\theta), \theta)$ and $(g(\theta), \theta)$. Write an expression for $w(\theta)$. Find w_A, the average value of $w(\theta)$ over the interval $0 \le \theta \le \frac{\pi}{2}$.

$w(\theta) = 2\cos\theta - 1 - \sin\theta\cos(2\theta)$

average value $= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} w(\theta)\,d\theta \approx 0.485$

(d) Using the information from part (c), find the value of θ for which $w(\theta) = w_A$. Is the function $w(\theta)$ increasing or decreasing at that value of θ? Give a reason for your answer.

$w(\theta) = 0.485461355 = B$

Calc intersect: Y is true when $\theta = 0.517$

0.51768795 $w'(\theta)$ when $\theta = 0.517$ is negative (-0.581)

Since $w'(\theta)$ is negative
then $w(\theta)$ is decreasing at $\theta = 0.517$

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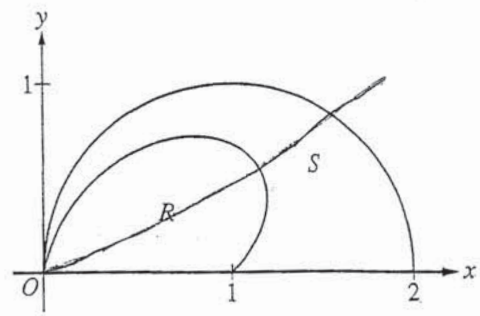
2B1



2. The figure above shows the polar curves $r = f(\theta) = 1 + \sin\theta\cos(2\theta)$ and $r = g(\theta) = 2\cos\theta$ for $0 \le \theta \le \frac{\pi}{2}$. Let R be the region in the first quadrant bounded by the curve $r = f(\theta)$ and the x-axis. Let S be the region in the first quadrant bounded by the curve $r = f(\theta)$, the curve $r = g(\theta)$, and the x-axis.

(a) Find the area of R.

$$\frac{1}{2}\int_0^{\pi/2} \left(1+\sin\theta\cos(2\theta)\right)^2 d\theta = 0.648$$

(b) The ray $\theta = k$, where $0 < k < \frac{\pi}{2}$, divides S into two regions of equal area. Write, but do not solve, an equation involving one or more integrals whose solution gives the value of k.

$$\boxed{\frac{1}{4}\int_0^{\pi/2} (2\cos\theta)^2 d\theta = \frac{1}{2}\int_0^{k} (2\cos\theta)^2 d\theta}$$

Continue question 2 on page 7.

2B2

(c) For each θ, $0 \le \theta \le \frac{\pi}{2}$, let $w(\theta)$ be the distance between the points with polar coordinates $(f(\theta), \theta)$ and $(g(\theta), \theta)$. Write an expression for $w(\theta)$. Find w_A, the average value of $w(\theta)$ over the interval $0 \le \theta \le \frac{\pi}{2}$.

$$w(\theta) = 2\cos\theta - 1 - \sin\theta\cos(2\theta)$$

$$\frac{\int_0^{\pi/2} (2\cos\theta - 1 - \sin\theta\cos(2\theta))\,d\theta}{\frac{\pi}{2} - 0} = \boxed{0.485}$$

(d) Using the information from part (c), find the value of θ for which $w(\theta) = w_A$. Is the function $w(\theta)$ increasing or decreasing at that value of θ ? Give a reason for your answer.

$$0.485 = 2\cos\theta - 1 - \sin\theta\cos(2\theta)$$

$$\theta = 0.518$$

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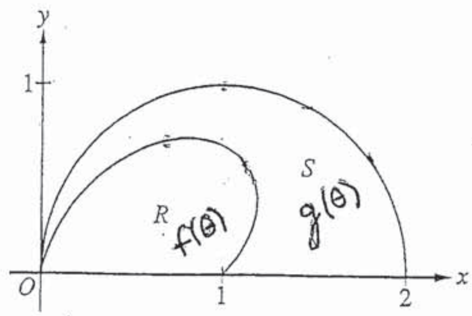
2C1



2. The figure above shows the polar curves $r = f(\theta) = 1 + \sin\theta\cos(2\theta)$ and $r = g(\theta) = 2\cos\theta$ for $0 \le \theta \le \frac{\pi}{2}$. Let R be the region in the first quadrant bounded by the curve $r = f(\theta)$ and the x-axis. Let S be the region in the first quadrant bounded by the curve $r = f(\theta)$, the curve $r = g(\theta)$, and the x-axis.

(a) Find the area of R.

$$\int_0^{\frac{\pi}{2}} \frac{1}{2}(1+\sin\theta\cos 2\theta)^2\, d\theta = .648$$

(b) The ray $\theta = k$, where $0 < k < \frac{\pi}{2}$, divides S into two regions of equal area. Write, but do not solve, an equation involving one or more integrals whose solution gives the value of k.

$$k = \frac{1}{2}\left[\int_0^{\frac{\pi}{2}} \frac{1}{2}(2\cos\theta)^2\, d\theta - \int_0^{\frac{\pi}{2}} \frac{1}{2}(1+\sin\theta\cos 2\theta)^2\, d\theta\right]$$

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Continue question 2 on page 7.

2C2

(c) For each θ, $0 \le \theta \le \frac{\pi}{2}$, let $w(\theta)$ be the distance between the points with polar coordinates $(f(\theta), \theta)$ and $(g(\theta), \theta)$. Write an expression for $w(\theta)$. Find w_A, the average value of $w(\theta)$ over the interval $0 \le \theta \le \frac{\pi}{2}$.

$w(\theta) = g(\theta) - f(\theta)$

$$\frac{1}{\frac{\pi}{2} - 0}\left[\int_0^{\frac{\pi}{2}} \frac{1}{2}(2\cos\theta)^2\,d\theta - \int_0^{\frac{\pi}{2}} \frac{1}{2}(1+\sin\theta\cos\theta)^2\,d\theta\right]$$

$= .359$

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(d) Using the information from part (c), find the value of θ for which $w(\theta) = w_A$. Is the function $w(\theta)$ increasing or decreasing at that value of θ ? Give a reason for your answer.

$.359 = (2\cos\theta - (1 + \sin\theta\cos\theta))$

$\theta =$

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Question 2

Overview

In this problem a polar graph is supplied for the curves $f(\theta) = 1 + \sin\theta\cos(2\theta)$ and $g(\theta) = 2\cos\theta$ for $0 \le \theta \le \frac{\pi}{2}$. Regions R, bounded by the graph of $r = f(\theta)$ and the x-axis, and region S, bounded by the graphs of $r = f(\theta)$, $r = g(\theta)$, and the x-axis, are identified on the graph. In part (a) students were asked for the area of R. Students needed to recognize that region R is traced by the polar ray segment from $r = 0$ to $r = f(\theta)$ for $0 \le \theta \le \frac{\pi}{2}$ and use the graphing calculator to evaluate the area of R as the numeric value of $\frac{1}{2}\int_0^{\pi/2}(f(\theta))^2\,d\theta$. [LO 3.4D/EK 3.4D1] In part (b) students were asked to produce an equation involving one or more integrals that can be solved for k, $0 < k < \frac{\pi}{2}$, such that the ray $\theta = k$ divides S into two regions of equal areas. Students needed to recognize that region S is traced by the polar ray segment from $r = f(\theta)$ to $r = g(\theta)$ for $0 \le \theta \le \frac{\pi}{2}$. The ray $\theta = k$ divides S into two subregions with areas $\frac{1}{2}\int_0^k\left((g(\theta))^2 - (f(\theta))^2\right)d\theta$ and $\frac{1}{2}\int_k^{\pi/2}\left((g(\theta))^2 - (f(\theta))^2\right)d\theta$. Students should have reported an equation equivalent to setting these two expressions equal to each other, or setting one of them equal to half of the area of S, which is given by $\frac{1}{2}\int_0^{\pi/2}\left((g(\theta))^2 - (f(\theta))^2\right)d\theta$. [LO 3.4D/EK 3.4D1] In part (c) $w(\theta)$ is defined as the distance between the points with polar coordinates $(f(\theta), \theta)$ and $(g(\theta), \theta)$. Students were asked to write an expression for $w(\theta)$ and to find w_A, the average value of $w(\theta)$ for $0 \le \theta \le \frac{\pi}{2}$. Students needed to recognize that $w(\theta) = g(\theta) - f(\theta)$ and use the graphing calculator to evaluate the average value $w_A = \dfrac{\int_0^{\pi/2} w(\theta)\,d\theta}{\frac{\pi}{2} - 0}$. [LO 3.4B/EK 3.4B1] In part (d) students were asked to find the value of θ for which $w(\theta) = w_A$, and to determine whether $w(\theta)$ is increasing or decreasing at that value of θ. Importing the value of w_A from part (c), students needed to use the graphing calculator to solve $w(\theta) = w_A$ to obtain $\theta = 0.517688$. Students should have reported this value rounded or truncated to three decimal places. Students should then have reported that $w(\theta)$ is decreasing at this value of θ because the calculator reports a negative value for $w'(0.517688)$. [LO 3.4D/EK 3.4D1] This problem incorporates the following Mathematical Practices for AP Calculus (MPACs): reasoning with definitions and theorems, connecting concepts, implementing algebraic/computational processes, connecting multiple representations, building notational fluency, and communicating.

Sample: 2A
Score: 9

The response earned all 9 points: 2 points in part (a), 2 points in part (b), 3 points in part (c), and 2 points in part (d). In part (a) the first point was earned for the integral $\int_0^{\frac{\pi}{2}} f(\theta)^2\,d\theta$. The second point was earned for the answer of 0.648. In part (b) the first point was earned by either $\int_0^k\left[g(\theta)^2 - f(\theta)^2\right]d\theta$ or

Question 2 (continued)

$\int_k^{\frac{\pi}{2}} \left[g(\theta)^2 - f(\theta)^2 \right] d\theta.$ The second point was earned for a correct equation using these two integral expressions. That is, the second point was earned for the student's equation $\frac{1}{2}\int_0^k \left[g(\theta)^2 - f(\theta)^2 \right] d\theta - \frac{1}{2}\int_k^{\frac{\pi}{2}} \left[g(\theta)^2 - f(\theta)^2 \right] d\theta = 0.$ In part (c) the first point was earned for the student's expression $w(\theta) = 2\cos\theta - 1 - \sin\theta\cos(2\theta).$ The second point was earned for the integral $\int_0^{\frac{\pi}{2}} w(\theta)\, d\theta.$ The third point was earned for the average value answer of 0.485. In part (d) the first point was earned for solving $w(\theta) = 0.4854461355$ to get $\theta = 0.517.$ The second point was earned for the student's conclusion that "Since $w'(\theta)$ is negative then $w(\theta)$ is decreasing at $\theta = 0.517.$"

Sample: 2B
Score: 6

The response earned 6 points: 2 points in part (a), no points in part (b), 3 points in part (c), and 1 point in part (d). In part (a) the first point was earned for the integral $\int_0^{\frac{\pi}{2}} (1 + \sin\theta\cos(2\theta))^2\, d\theta.$ The second point was earned for the answer of 0.648. In part (b) the first point was not earned because the student does not present a correct integral expression with an integrand in the form $(g(\theta))^2 - (f(\theta))^2.$ The student's integral expressions only involve $g(\theta)$ and do not involve $f(\theta).$ Because the first point was not earned, the student is not eligible for the second point. In part (c) the first point was earned for the student's expression $w(\theta) = 2\cos\theta - 1 - \sin\theta\cos(2\theta).$ The second point was earned for the integral $\int_0^{\frac{\pi}{2}} (2\cos\theta - 1 - \sin\theta\cos(2\theta))\, d\theta.$ The third point was earned for the average value answer of 0.485. In part (d) the first point was earned for solving $w(\theta) = 0.485$ to get $\theta = 0.518.$ Because the student does not provide a conclusion for whether $w(\theta)$ is increasing or decreasing, the second point was not earned.

Sample: 2C
Score: 3

The response earned 3 points: 2 points in part (a), no points in part (b), 1 point in part (c), and no points in part (d). In part (a) the first point was earned for the integral $\int_0^{\frac{\pi}{2}} (1 + \sin\theta\cos 2\theta)^2\, d\theta.$ The second point was earned for the answer of .648. In part (b) the first point was not earned because the student does not present a correct integral expression where one of the limits of integration is k. The student is not eligible to earn the second point without earning the first point. In part (c) the first point was earned for the expression $w(\theta) = g(\theta) - f(\theta).$ The second point was not earned because the student's integral expression $\int_0^{\frac{\pi}{2}} \frac{1}{2}(2\cos\theta)^2\, d\theta - \int_0^{\frac{\pi}{2}} \frac{1}{2}(1 + \sin\theta\cos\theta)^2\, d\theta$ is incorrect. The third point was not earned because the student's answer of .359 is incorrect. The third point can only be earned with the average value answer 0.485. In part (d) the first point was not earned because the student does not solve $w(\theta) = .359.$ Without a value for $\theta,$ the student is not eligible for the second point.

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AP Calculus BC

Sample Student Responses and Scoring Commentary

Inside:

- ☑ Free Response Question 3
- ☑ Scoring Guideline
- ☑ Student Samples
- ☑ Scoring Commentary

Question 3

(a) $f(-6) = f(-2) + \int_{-2}^{-6} f'(x)\,dx = 7 - \int_{-6}^{-2} f'(x)\,dx = 7 - 4 = 3$

$f(5) = f(-2) + \int_{-2}^{5} f'(x)\,dx = 7 - 2\pi + 3 = 10 - 2\pi$

$3 : \begin{cases} 1 : \text{uses initial condition} \\ 1 : f(-6) \\ 1 : f(5) \end{cases}$

(b) $f'(x) > 0$ on the intervals $[-6, -2)$ and $(2, 5)$.
Therefore, f is increasing on the intervals $[-6, -2]$ and $[2, 5]$.

2 : answer with justification

(c) The absolute minimum will occur at a critical point where $f'(x) = 0$ or at an endpoint.

$f'(x) = 0 \Rightarrow x = -2, x = 2$

x	$f(x)$
-6	3
-2	7
2	$7 - 2\pi$
5	$10 - 2\pi$

The absolute minimum value is $f(2) = 7 - 2\pi$.

$2 : \begin{cases} 1 : \text{considers } x = 2 \\ 1 : \text{answer with justification} \end{cases}$

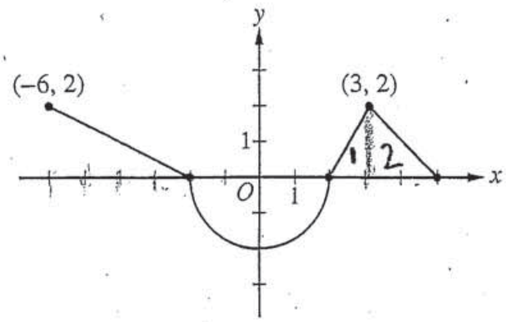
(d) $f''(-5) = \dfrac{2 - 0}{-6 - (-2)} = -\dfrac{1}{2}$

$\lim\limits_{x \to 3^-} \dfrac{f'(x) - f'(3)}{x - 3} = 2$ and $\lim\limits_{x \to 3^+} \dfrac{f'(x) - f'(3)}{x - 3} = -1$

$f''(3)$ does not exist because
$\lim\limits_{x \to 3^-} \dfrac{f'(x) - f'(3)}{x - 3} \neq \lim\limits_{x \to 3^+} \dfrac{f'(x) - f'(3)}{x - 3}.$

$2 : \begin{cases} 1 : f''(-5) \\ 1 : f''(3) \text{ does not exist,} \\ \quad\ \text{with explanation} \end{cases}$

3A, 3A,



Graph of f'

3. The function f is differentiable on the closed interval $[-6, 5]$ and satisfies $f(-2) = 7$. The graph of f', the derivative of f, consists of a semicircle and three line segments, as shown in the figure above.

(a) Find the values of $f(-6)$ and $f(5)$.

$$f(-6) = \left(\int_{-2}^{-6} f'(x)\,dx\right) + f(-2)$$

$$f(-6) = 3$$

$$f(5) = f(-2) + \int_{-2}^{5} f'(x)\,dx$$

$$f(5) = 10 - 2\pi$$

(b) On what intervals is f increasing? Justify your answer.

f is increasing on $x = [-6, -2] \cup [2, 5]$, since $f' > 0$ on the interval $x \in [-6, -2) \cup (2, 5)$

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Continue question 3 on page 15.

3A₂

NO CALCULATOR ALLOWED

3A₂

(c) Find the absolute minimum value of f on the closed interval $[-6, 5]$. Justify your answer.

The absolute minimum of f on $[-6,5]$ is $7-2\pi$, since $f(2) < f(5)$ and $f(-6)$ (the endpoints) and $f(2) < f(-2)$ the other critical point, by EVT

Endpoints
$f(-6) = 3$
$f(5) = 10-2\pi$
critical points
$f' = 0$
$f(-2) = 7$
$f(2) = 7-2\pi$

(d) For each of $f''(-5)$ and $f''(3)$, find the value or explain why it does not exist.

$$f''(-5) = \frac{-1}{2}$$

$f''(3) =$ DNE, as the

$$\lim_{x \to 3^+} \frac{f'(x)-2}{x-3} \neq \lim_{x \to 3^-} \frac{f'(x)-2}{x-3}$$

Therefore it is impossible to take a derivative at $x=3$ in f'

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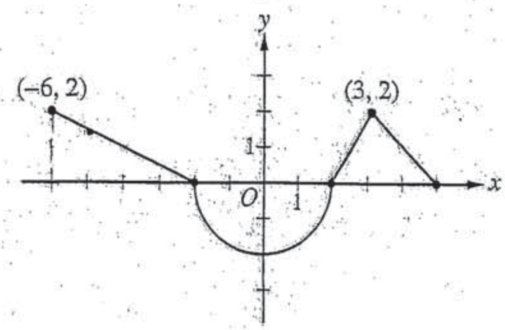

Graph of f'

3. The function f is differentiable on the closed interval $[-6, 5]$ and satisfies $f(-2) = 7$. The graph of f', the derivative of f, consists of a semicircle and three line segments, as shown in the figure above.

(a) Find the values of $f(-6)$ and $f(5)$.

$f(-6) = f(-2) - \int_{-6}^{-2} f'(x)\,dx$

$f(-6) = 7 - \frac{4 \times 2}{2} = \boxed{3}$

$f(5) = f(-2) + \int_{-2}^{5} f'(x)\,dx$

$f(5) = 7 + \frac{3 \times 2}{2} - \frac{1}{2}\pi \times 2^2 = \boxed{10 - 2\pi}$

(b) On what intervals is f increasing? Justify your answer.

Since on intervals of $(-6, 2)$ and $(2, 5)$; $f'(x) > 0$

then $f(x)$ is increasing on intervals $[-6, 2]$ and $[2, 5]$

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Continue question 3 on page 15.

(c) Find the absolute minimum value of f on the closed interval $[-6, 5]$. Justify your answer.

$f(x)$ has its absolute minimum on either two endpoints and where $f'(x)=0$

according to the graph : $f'(-2) = f'(2) = 0$

x	$f(x)$
-6	3
-2	7
2	$7-2\pi$
5	$10-2\pi$

according to the table, $f(x)$ reaches its absolute minimum value $\boxed{7-2\pi}$ at $x=2$

(d) For each of $f''(-5)$ and $f''(3)$, find the value or explain why it does not exist.

$f''(-5) = \frac{d}{dx} f'(x)\Big|_{x=-5} = \frac{2}{-4} = \boxed{-\frac{1}{2}}$

Since $\lim_{x\to 3^-} f''(x) \neq \lim_{x\to 3^+} f''(x)$

then $f(x)$ is not differentiable at $x=3$

therefore, $f''(3)$ does not exist.

3C, 3C,

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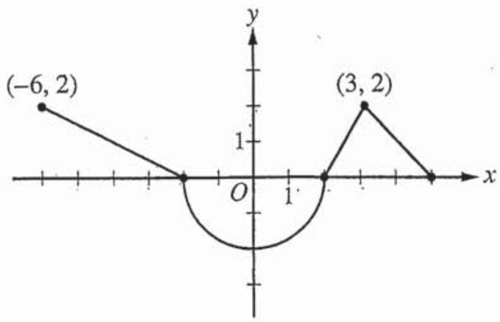

Graph of f'

3. The function f is differentiable on the closed interval $[-6, 5]$ and satisfies $f(-2) = 7$. The graph of f', the derivative of f, consists of a semicircle and three line segments, as shown in the figure above.

(a) Find the values of $f(-6)$ and $f(5)$.

$f(-6) = \frac{2}{5} \times (2) = \frac{4}{5}$

$f(5) = 0$

(b) On what intervals is f increasing? Justify your answer.

From $[-6, -2]$ and $[2, 5]$, f is increasing because the graph of $f'(x)$ is >0 from $([-6, -2)$ and $(2, 5)$

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Continue question 3 on page 15.

3C2 3C2

NO CALCULATOR ALLOWED

(c) Find the absolute minimum value of f on the closed interval $[-6, 5]$. Justify your answer.

The absolute minimum value of 0 at x = 2.
because the graph of f' changes sign from negative to positive at x = 2.

(d) For each of $f''(-5)$ and $f''(3)$, find the value or explain why it does not exist.

f"(3) is an inflection point because f' increases on [2, 3] and decreases on [2, 4].
f"(-5) does not exist because the graph of f' from [-6, -2] has a slope of $\frac{2}{5}$.

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GO ON TO THE NEXT PAGE.

Question 3

Overview

In this problem students were given that a function f is differentiable on the interval $[-6, 5]$ and satisfies $f(-2) = 7$. For $-6 \le x \le 5$, the derivative of f is specified by a graph consisting of a semicircle and three line segments. In part (a) students were asked to find values of $f(-6)$ and $f(5)$. For each of these values, students needed to recognize that the net change in f, starting from the given value $f(-2) = 7$, can be computed using a definite integral of $f'(x)$ with a lower limit of integration -2 and an upper limit the desired argument of f. These integrals can be computed using properties of the definite integral and the geometric connection to areas between the graph of $y = f'(x)$ and the x-axis. Thus, students needed to add the initial condition $f(-2) = 7$ to the values of the definite integrals for the desired values. [LO 3.2C/EK 3.2C1] In part (b) students were asked for the intervals on which f is increasing, with justification. Since f' is given on the interval $[-6, 5]$, f is differentiable, and thus also continuous, on that interval. Therefore, f is increasing on closed intervals for which $f'(x) > 0$ on the interior. Students needed to use the given graph of f' to see that $f'(x) > 0$ on the intervals $[-6, -2)$ and $(2, 5)$, so f is increasing on the intervals $[-6, -2]$ and $[2, 5]$, connecting their answers to the sign of f'. [LO 2.2A/EK 2.2A1-2.2A2, LO 2.2B/EK 2.2B1] In part (c) students were asked for the absolute minimum value of f on the closed interval $[-6, 5]$, and to justify their answers. Students needed to use the graph of f' to identify critical points of f on the interior of the interval as $x = -2$ and $x = 2$. Then they can compute $f(-2)$ and $f(2)$, similarly to the computations in part (a), and compare these to the values of f at the endpoints that were computed in part (a). Students needed to report the smallest of these values, $f(2) = 7 - 2\pi$ as the answer. Alternatively, students could have observed that the minimum value must occur either at a point interior to the interval at which f' transitions from negative to positive, at a left endpoint for which f' is positive immediately to the right, or at a right endpoint for which f' is negative immediately to the left. This reduces the options to $f(-6) = 3$ and $f(2) = 7 - 2\pi$. [LO 2.2A/EK 2.2A1-2.2A2, LO 2.2B/EK 2.2B1, LO 3.3A/EK 3.3A3] In part (d) students were asked to determine values of $f''(-5)$ and $f''(3)$, or to explain why the requested value does not exist. Students needed to find the value $f''(-5)$ as the slope of the line segment on the graph of f' through the point corresponding to $x = -5$. The point on the graph of f' corresponding to $x = 3$ is the juncture of a line segment of slope 2 on the left with one of slope -1 on the right. Thus, students needed to report that $f''(3)$ does not exist, and explain why the given graph of f' shows that f' is not differentiable at $x = 3$. Student explanations could be done by noting that the left-hand and right-hand limits at $x = 3$ of the difference quotient $\frac{f'(x) - f'(3)}{x - 3}$ have differing values (2 and -1, respectively), or by a clear description of the relevant features of the graph of f' near $x = 3$. [LO 1.1A(b)/EK 1.1A3] This problem incorporates the following Mathematical Practices for AP Calculus (MPACs): reasoning with definitions and theorems, connecting concepts, implementing algebraic/computational processes, connecting multiple representations, building notational fluency, and communicating.

Sample: 3A
Score: 9

The response earned all 9 points: 3 points in part (a), 2 points in part (b), 2 points in part (c), and 2 points in part (d). In part (a) the student uses the initial condition $f(-2)$ with an appropriate definite integral $\int_{-2}^{-6} f'(x)\,dx$ to find $f(-6) = 3$. Thus, the student earned the first and second points. The student uses $f(-2)$ again with an

Question 3 (continued)

appropriate definite integral $\int_{-2}^{5} f'(x)\,dx$ to find $f(5) = 10 - 2\pi$. The student earned the third point. In part (b) the student states two correct and complete intervals, $[-6, -2]$ and $[2, 5]$, where f is increasing. The student justifies the intervals with a discussion of $f' > 0$ for $[-6, -2)$ and $(2, 5)$. The student earned both points. In part (c) the student considers $x = -6, -2, 2,$ and 5 as potential locations for the absolute minimum value. The student earned the first point for considering $x = 2$. The student identifies the absolute minimum value as $7 - 2\pi$. The student justifies by evaluating $f(x)$ at the critical values and endpoints. The student earned the second point. In part (d) the student finds $f''(-5) = -\frac{1}{2}$ and earned the first point. The student states that $f''(3)$ does not exist. The student uses two one-sided limits at $x = 3$ to explain why the derivative of $f'(x)$ does not exist and earned the second point.

Sample: 3B
Score: 6

The response earned 6 points: 3 points in part (a), no points in part (b), 2 points in part (c), and 1 point in part (d). In part (a) the student uses the initial condition $f(-2)$ with an appropriate definite integral $\int_{-6}^{-2} f'(x)\,dx$ to find $f(-6) = 3$. Thus, the student earned the first and second points. The student uses $f(-2)$ again with an appropriate definite integral $\int_{-2}^{5} f'(x)\,dx$ to find $f(5) = 10 - 2\pi$. The student earned the third point. In part (b) the student presents two intervals, $[-6, 2)$ and $(2, 5)$. Because $f'(x) < 0$ on $(-2, 2)$, f is decreasing on $[-2, 2]$. The student is not eligible to earn any points because of the presence of an interval containing points where $f'(x) < 0$. Thus, the student did not earn any points. In part (c) the student investigates where $f'(x) = 0$ and identifies $f'(-2)$ and $f'(2)$. The student earned the first point for considering $x = 2$. The student identifies the absolute minimum value as $7 - 2\pi$. The student justifies by evaluating $f(x)$ at the critical values and endpoints. The student earned the second point. In part (d) the student identifies $f''(-5)$ as the derivative of $f'(x)$ at $x = -5$ and finds $f''(-5) = -\frac{1}{2}$. The student earned the first point. The student states that $f''(3)$ does not exist. The student uses two one-sided limits at $x = 3$. The student states that "$f(x)$ is not differentiable at $x = 3$," which contradicts the given statement in the problem that f is differentiable on the closed interval $[-6, 5]$. The student did not earn the second point.

Sample: 3C
Score: 3

The response earned 3 points: no points in part (a), 2 points in part (b), 1 point in part (c), and no points in part (d). In part (a) the student never uses the initial condition, incorrectly evaluates $f(-6)$ as $\frac{4}{5}$, and incorrectly evaluates $f(5)$ as 0. The student earned no points. In part (b) the student states two correct and complete intervals, $[-6, -2]$ and $[2, 5]$, on which f is increasing. The student justifies the intervals with "$f'(x)$ is > 0 from $[-6, -2)$ and $(2, 5)$." The student earned both points. In part (c) the student considers $x = 2$ and earned the first point. The student presents an incorrect answer for the absolute minimum value with an incorrect justification. The student does not evaluate $f(x)$ at the critical values and endpoints in order to determine the

Question 3 (continued)

absolute minimum value. The student did not earn the second point. In part (d) the student incorrectly determines that $f''(-5)$ has a value of $\frac{2}{5}$ and did not earn the first point. The student states that "$f''(3)$ is an inflection point" and does not state that $f''(3)$ does not exist. The student did not earn the second point.

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Sample Student Responses and Scoring Commentary

Inside:

- ☑ Free Response Question 4
- ☑ Scoring Guideline
- ☑ Student Samples
- ☑ Scoring Commentary

Question 4

(a) $H'(0) = -\frac{1}{4}(91 - 27) = -16$ $H(0) = 91$ An equation for the tangent line is $y = 91 - 16t$. The internal temperature of the potato at time $t = 3$ minutes is approximately $91 - 16 \cdot 3 = 43$ degrees Celsius.	$3 : \begin{cases} 1 : \text{slope} \\ 1 : \text{tangent line} \\ 1 : \text{approximation} \end{cases}$
(b) $\frac{d^2H}{dt^2} = -\frac{1}{4}\frac{dH}{dt} = \left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)(H - 27) = \frac{1}{16}(H - 27)$ $H > 27$ for $t > 0 \Rightarrow \frac{d^2H}{dt^2} = \frac{1}{16}(H - 27) > 0$ for $t > 0$ Therefore, the graph of H is concave up for $t > 0$. Thus, the answer in part (a) is an underestimate.	1 : underestimate with reason
(c) $\frac{dG}{(G - 27)^{2/3}} = -dt$ $\int \frac{dG}{(G - 27)^{2/3}} = \int (-1)\, dt$ $3(G - 27)^{1/3} = -t + C$ $3(91 - 27)^{1/3} = 0 + C \Rightarrow C = 12$ $3(G - 27)^{1/3} = 12 - t$ $G(t) = 27 + \left(\frac{12 - t}{3}\right)^3$ for $0 \le t < 10$ The internal temperature of the potato at time $t = 3$ minutes is $27 + \left(\frac{12 - 3}{3}\right)^3 = 54$ degrees Celsius.	$5 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration and} \\ \quad \text{uses initial condition} \\ 1 : \text{equation involving } G \text{ and } t \\ 1 : G(t) \text{ and } G(3) \end{cases}$ Note: max $2/5$ [1-1-0-0-0] if no constant of integration Note: $0/5$ if no separation of variables

NO CALCULATOR ALLOWED

4A₁

4. At time $t = 0$, a boiled potato is taken from a pot on a stove and left to cool in a kitchen. The internal temperature of the potato is 91 degrees Celsius (°C) at time $t = 0$, and the internal temperature of the potato is greater than 27°C for all times $t > 0$. The internal temperature of the potato at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{4}(H - 27)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 91$.

(a) Write an equation for the line tangent to the graph of H at $t = 0$. Use this equation to approximate the internal temperature of the potato at time $t = 3$.

$\left.\frac{dH}{dt}\right|_{H=91} = -\frac{1}{4}(91-27) = \left(-\frac{1}{4}\right)(64) = -16$

$A(t) = -16(t-0) + 91$

$\boxed{A(t) = -16t + 91}$

$A(3) = -16(3) + 91$
$= -48 + 91$
$= \boxed{43°C \text{ at } t = 3 \text{ minutes}}$

(b) Use $\frac{d^2H}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the internal temperature of the potato at time $t = 3$.

$\frac{dH}{dt} = -\frac{1}{4}(H-27)$

$\frac{d^2H}{dt^2} = -\frac{1}{4}\left(\frac{dH}{dt}\right)$

$= -\frac{1}{4}\left(-\frac{1}{4}\right)(H-27)$

$\frac{d^2H}{dt^2} = \frac{1}{16}(H-27)$

$H > 27$ for all $t > 0$

$\frac{d^2H}{dt^2}$ is always positive, so part (a) is an $\boxed{\text{underestimate.}}$

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Continue question 4 on page 17.

NO CALCULATOR ALLOWED

4A$_2$

(c) For $t < 10$, an alternate model for the internal temperature of the potato at time t minutes is the function G that satisfies the differential equation $\frac{dG}{dt} = -(G-27)^{2/3}$, where $G(t)$ is measured in degrees Celsius and $G(0) = 91$. Find an expression for $G(t)$. Based on this model, what is the internal temperature of the potato at time $t = 3$?

$$\frac{dG}{dt} = -(G-27)^{2/3}$$

$$\int \frac{dG}{(G-27)^{2/3}} = \int -1\,dt$$

$$\int (G-27)^{-2/3}\,dG = -t + C$$

$$3(G-27)^{1/3} = -t + C_1$$

$$(G-27)^{1/3} = -\frac{t}{3} + C_2$$

$$G-27 = \left(-\frac{t}{3} + C_2\right)^3$$

$$G = \left(-\frac{t}{3} + C_2\right)^3 + 27$$

at $(0, 91)$

$$91 = (0 + C_2)^3 + 27$$

$$91 = (C_2)^3 + 27$$

$$64 = (C_2)^3 \rightarrow C_2 = 4$$

91
27

$$\boxed{G(t) = \left(-\frac{t}{3} + 4\right)^3 + 27}$$

$$G(3) = \left(-\frac{3}{3} + 4\right)^3 + 27$$

$$= (3)^3 + 27$$

$$= 27 + 27$$

$$\boxed{G(3) = 54°C}$$

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4B1

4. At time $t = 0$, a boiled potato is taken from a pot on a stove and left to cool in a kitchen. The internal temperature of the potato is 91 degrees Celsius (°C) at time $t = 0$, and the internal temperature of the potato is greater than 27°C for all times $t > 0$. The internal temperature of the potato at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{4}(H - 27)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 91$.

(a) Write an equation for the line tangent to the graph of H at $t = 0$. Use this equation to approximate the internal temperature of the potato at time $t = 3$.

$$\frac{dH}{dt} = -\frac{1}{4}(91-27)$$

$$\frac{dH}{dt} = -\frac{1}{4}(64)$$

$$\frac{dH}{dt} = -16$$

eauation of tangent line:

$$y - 91 = -16t$$

$$y = -16t + 91$$

approximation at time $t=3$:

$$y = -16(3) + 91$$

$$y = -48 + 91$$

$$= 43°C$$

at time $t=3$

(b) Use $\frac{d^2H}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the internal temperature of the potato at time $t = 3$.

$$\frac{dH'}{dt} = -\frac{1}{4}(H-27)$$

$$\frac{dH}{dt} = -\frac{1}{4}H + \frac{27}{4}$$

$$\frac{d^2H}{dt^2} = -\frac{1}{4}$$

underestimate because the value is less that the estimated value of $\frac{d^2H}{dt^2}$ at $t=3$

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Continue question 4 on page 17.

4B$_2$

(c) For $t < 10$, an alternate model for the internal temperature of the potato at time t minutes is the function G that satisfies the differential equation $\frac{dG}{dt} = -(G - 27)^{2/3}$, where $G(t)$ is measured in degrees Celsius and $G(0) = 91$. Find an expression for $G(t)$. Based on this model, what is the internal temperature of the potato at time $t = 3$?

$$\frac{1}{-(G-27)^{2/3}}\,dG = 1\,dt$$

$$3(G-27)^{1/3} = t + C$$

$$3(91-27)^{1/3} = 0 + C$$

$$3(64)^{1/3} = C$$

$$3(4) = C$$

$$12 = C$$

$$3(G-27)^{1/3} = t + 12$$

$$(G-27)^{\cancel{1/3}} = \left(\frac{t+12}{3}\right)^3 + 27$$

$$G(t) = \left(\frac{t+12}{3}\right)^3 + 27$$

$$G(3) = \left(\frac{15}{3}\right)^3 + 27$$

$$G(3) = (5)^3 + 27$$

$$G(3) = 125 + 27$$

$$G(3) = 152$$

152°C at time $t = 3$

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GO ON TO THE NEXT PAGE.

4C1

4. At time $t = 0$, a boiled potato is taken from a pot on a stove and left to cool in a kitchen. The internal temperature of the potato is 91 degrees Celsius (°C) at time $t = 0$, and the internal temperature of the potato is greater than 27°C for all times $t > 0$. The internal temperature of the potato at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{4}(H - 27)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 91$.

(a) Write an equation for the line tangent to the graph of H at $t = 0$. Use this equation to approximate the internal temperature of the potato at time $t = 3$.

$\frac{dH}{dt} = -\frac{1}{4}(91-27)$

$= -\frac{1}{4}(64) = -16$

$H - 91 = -16(t)$

$H = -16t + 91$

$H = -16(3) + 91$

$H = -48 + 91$

$H = 43°C$

(b) Use $\frac{d^2H}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the internal temperature of the potato at time $t = 3$.

$\frac{dH}{dt} = -\frac{1}{4}H + \frac{27}{4}$

$\frac{d^2H}{dt^2} = -\frac{1}{4}$

This is an underestimate because $\frac{dH}{dt}$ + $\frac{d^2H}{dt^2}$ are both decreasing

Continue question 4 on page 17.

NO CALCULATOR ALLOWED

4C2

(c) For $t < 10$, an alternate model for the internal temperature of the potato at time t minutes is the function G that satisfies the differential equation $\frac{dG}{dt} = -(G-27)^{2/3}$, where $G(t)$ is measured in degrees Celsius and $G(0) = 91$. Find an expression for $G(t)$. Based on this model, what is the internal temperature of the potato at time $t = 3$?

$\frac{dG}{dt} = -(G-27)^{\frac{2}{3}}$

$\int \frac{dG}{dt} = \int -(G-27)^{\frac{2}{3}} dt$

$G(t) = -\int (G-27)^{\frac{2}{3}} dt$ $\quad u = G-27$

$du = dt$

$G(t) = -\int u^{\frac{2}{3}} du$

$G(t) = -\frac{u^{\frac{5}{3}}}{\frac{5}{3}}$ $\boxed{-\frac{3}{5}(G-27)^{\frac{5}{3}}}$

$-\frac{3}{5}(91-27)^{\frac{5}{3}}$

$-\frac{3}{5}(64)^{\frac{5}{3}}$

64
$\times 4$
256
$\times 4$
1024

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GO ON TO THE NEXT PAGE.

Question 4

Overview

The context for this problem is the internal temperature of a boiled potato that is left to cool in a kitchen. Initially at time $t = 0,$ the potato's internal temperature is 91 degrees Celsius, and it is given that the internal temperature of the potato exceeds 27 degrees Celsius for all times $t > 0$. The internal temperature of the potato at time t minutes is modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{4}(H - 27),$ where $H(t)$ is measured in degrees Celsius and $H(0) = 91.$ In part (a) students were asked for an equation of the line tangent to the graph of H at $t = 0,$ and to use this equation to approximate the internal temperature of the potato at time $t = 3.$ Using the initial value and the differential equation, students needed to find the slope of the tangent line to be $H'(0) = -\frac{1}{4}(91 - 27) = -16$ and report the equation of the tangent line to be $y = 91 - 16t.$ Students needed to find the approximate temperature of the potato at $t = 3$ to be $91 - 16 \cdot 3 = 43$ degrees Celsius. [LO 2.3B/EK 2.3B2] In part (b) students were asked to use $\frac{d^2H}{dt^2}$ to determine whether the approximation in part (a) is an underestimate or overestimate for the potato's internal temperature at time $t = 3.$ Students needed to use the given differential equation to calculate $\frac{d^2H}{dt^2} = -\frac{1}{4}\frac{dH}{dt} = \frac{1}{16}(H - 27).$ Then using the given information that the temperature always exceeds 27 degrees Celsius, students needed to conclude that $\frac{d^2H}{dt^2} > 0$ for all times $t.$ Thus, the graph of H is concave up, and the line tangent to the graph of H at $t = 0$ lies below the graph of H (except at the point of tangency), so the approximation found in part (a) is an underestimate. [LO 2.1D/EK 2.1D1, LO 2.2A/EK 2.2A1] In part (c) an alternate model, G, is proposed for the internal temperature of the potato at times $t < 10.$ $G(t)$ is measured in degrees Celsius and satisfies the differential equation $\frac{dG}{dt} = -(G - 27)^{2/3}$ with $G(0) = 91.$ Students were asked to find an expression for $G(t)$ and to find the internal temperature of the potato at time $t = 3$ based on this model. Students needed to employ the method of separation of variables, using the initial condition $G(0) = 91$ to resolve the constant of integration, and arrive at the particular solution $G(t) = 27 + \left(\frac{12 - t}{3}\right)^3$. Students should then have reported that the model gives an internal temperature of $G(3) = 54$ degrees Celsius for the potato at time $t = 3.$ [LO 3.5A/EK 3.5A2] This problem incorporates the following Mathematical Practices for AP Calculus (MPACs): reasoning with definitions and theorems, connecting concepts, implementing algebraic/computational processes, building notational fluency, and communicating.

Sample: 4A
Score: 9

The response earned all 9 points: 3 points in part (a), 1 point in part (b), and 5 points in part (c). In part (a) the student earned the first point for the slope with $-\frac{1}{4}(91 - 27).$ The second point was earned for the tangent line $A(t) =-16(t - 0) + 91.$ Note that the student names the tangent line $A(t)$ and is not penalized. The third point was earned for the approximation 43. Either of the numerical expressions $-16(3) + 91$ or $-48 + 91$ would have earned the third point. The student chooses to simplify and does so correctly. In part (b) the student has the correct answer of "underestimate" and supports the answer with correct reasoning. The student has the correct second derivative in line 2 on the left and states that the second derivative is always positive in line 2 on the right. The

Question 4 (continued)

student earned the point. In part (c) the student earned the first point for separation in line 2. The antiderivatives are correct in line 4 and earned the second point. The third point was earned for the constant of integration and use of the initial condition in lines 8 and 9. The fourth point was earned in line 12 for an equation involving G and t and a correct numerical value for C. As an aside, the fourth point could have been earned with an implicit equation such as $(G-27)^{1/3} = \frac{12-t}{3}$. The fifth point was earned for $G(t) = \left(-\frac{t}{3}+4\right)^3 + 27$ in line 12 along with 54 in line 4 on the right. Any of the three numerical expressions in lines 1, 2, and 3 on the right together with $G(t) = \left(-\frac{t}{3}+4\right)^3 + 27$ in line 12 would have earned the fifth point. The student chooses to simplify and does so correctly.

Sample: 4B
Score: 6

The response earned 6 points: 3 points in part (a), no point in part (b), and 3 points in part (c). In part (a) the student earned the first point for the slope with $-\frac{1}{4}(91-27)$. The second point was earned for the tangent line $y - 91 = -16t$. The third point was earned for the approximation 43. Either of the numerical expressions $-16(3)+91$ or $-48+91$ would have earned the third point. The student chooses to simplify and does so correctly. In part (b) the student's answer of "underestimate" is correct, but the reason is based on an incorrect second derivative in line 3 on the left. The student did not earn the point. In part (c) the student earned the first point for separation in line 1. In line 2 the student drops a negative sign, so the second point for correct antiderivatives was not earned. Line 2 should have been: $-3(G-27)^{1/3} = t + C$. The student's equation involving antiderivatives is eligible for the remaining 3 points. In line 3 the third point was earned for the constant of integration and use of the initial condition. In line 7 the fourth point was earned for an equation involving G and t together with the consistent numerical value for C. The fifth point was not earned because, although the answer is consistent with the work, the student's value of $G(3)$ is out of the context of the problem because $152 > 91$.

Sample: 4C
Score: 3

The response earned 3 points: 3 points in part (a), no point in part (b), and no points in part (c). In part (a) the student earned the first point for the slope with $-\frac{1}{4}(91-27)$ in line 1 on the left. The second point was earned for the tangent line $H - 91 = -16(t)$. Note that the student uses H in place of y in the tangent line and is not penalized. The third point was earned for the approximation 43. Either of the numerical expressions $-16(3)+91$ or $-48+91$ would have earned the third point. In part (b) the student's answer of "underestimate" is correct, but the reason is based on an incorrect second derivative of $-\frac{1}{4}$. The student did not earn the point. In part (c) there is no separation of variables, and thus, no points were earned.

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AP Calculus BC

Sample Student Responses and Scoring Commentary

Inside:

- ☑ Free Response Question 5
- ☑ Scoring Guideline
- ☑ Student Samples
- ☑ Scoring Commentary

Question 5

(a) $f'(x) = \dfrac{-3(4x-7)}{\left(2x^2-7x+5\right)^2}$ $f'(3) = \dfrac{(-3)(5)}{(18-21+5)^2} = -\dfrac{15}{4}$	2 : $f'(3)$
(b) $f'(x) = \dfrac{-3(4x-7)}{\left(2x^2-7x+5\right)^2} = 0 \Rightarrow x = \dfrac{7}{4}$ The only critical point in the interval $1 < x < 2.5$ has x-coordinate $\dfrac{7}{4}$. f' changes sign from positive to negative at $x = \dfrac{7}{4}$. Therefore, f has a relative maximum at $x = \dfrac{7}{4}$.	2 : $\begin{cases} 1 : x\text{-coordinate} \\ 1 : \text{relative maximum} \\ \quad\ \text{with justification} \end{cases}$
(c) $\displaystyle\int_5^\infty f(x)\,dx = \lim_{b\to\infty}\int_5^b \frac{3}{2x^2-7x+5}\,dx = \lim_{b\to\infty}\int_5^b \left(\frac{2}{2x-5} - \frac{1}{x-1}\right)dx$ $\displaystyle = \lim_{b\to\infty}\Big[\ln(2x-5) - \ln(x-1)\Big]_5^b = \lim_{b\to\infty}\left[\ln\left(\frac{2x-5}{x-1}\right)\right]_5^b$ $\displaystyle = \lim_{b\to\infty}\left[\ln\left(\frac{2b-5}{b-1}\right) - \ln\left(\frac{5}{4}\right)\right] = \ln 2 - \ln\left(\frac{5}{4}\right) = \ln\left(\frac{8}{5}\right)$	3 : $\begin{cases} 1 : \text{antiderivative} \\ 1 : \text{limit expression} \\ 1 : \text{answer} \end{cases}$
(d) f is continuous, positive, and decreasing on $[5, \infty)$. The series converges by the integral test since $\displaystyle\int_5^\infty \frac{3}{2x^2-7x+5}\,dx$ converges. — OR — $\dfrac{3}{2n^2-7n+5} > 0$ and $\dfrac{1}{n^2} > 0$ for $n \ge 5$. Since $\displaystyle\lim_{n\to\infty}\frac{\frac{3}{2n^2-7n+5}}{\frac{1}{n^2}} = \frac{3}{2}$ and the series $\displaystyle\sum_{n=5}^\infty \frac{1}{n^2}$ converges, the series $\displaystyle\sum_{n=5}^\infty \frac{3}{2n^2-7n+5}$ converges by the limit comparison test.	2 : answer with conditions

NO CALCULATOR ALLOWED

5A$_1$

5. Let f be the function defined by $f(x) = \dfrac{3}{2x^2 - 7x + 5}$.

(a) Find the slope of the line tangent to the graph of f at $x = 3$.

$f(x) = 3(2x^2 - 7x + 5)^{-1}$

$f'(x) = -3(2x^2 - 7x + 5)^{-2}(4x - 7)$

$f'(3) = \dfrac{-3(4(3)-7)}{(2(3)^2 - 7(3) + 5)^2} = \dfrac{-15}{(18 - 21 + 5)^2} = \dfrac{-15}{4}$

(b) Find the x-coordinate of each critical point of f in the interval $1 < x < 2.5$. Classify each critical point as the location of a relative minimum, a relative maximum, or neither. Justify your answers.

$0 = f'(x)$

$0 = \dfrac{-3(4x-7)}{(2x^2 - 7x + 5)^2}$

$(2x-5)(2x-1)$

$\rightarrow 0 = -3(4x - 7)$

$x = \frac{7}{4}$

$f'(1.5) \quad f'(2)$

$+ \quad \frac{7}{4} \quad -$

$\rightarrow 0 = (2x^2 - 7x + 5)^2$

$0 = (2x^2 - 7x + 5)$

$0 = (2x - 5)(x - 1)$

$\cancel{x = \frac{5}{2}} \quad \cancel{x = 1}$

$x = \frac{7}{4}$ is a relative maximum because $f'(x)$ goes from positive to negative

Continue question 5 on page 19.

5A2

(c) Using the identity that $\frac{3}{2x^2 - 7x + 5} = \frac{2}{2x - 5} - \frac{1}{x - 1}$, evaluate $\int_5^\infty f(x)\,dx$ or show that the integral diverges.

$$\int_5^\infty f(x)dx = \int_5^\infty \frac{2}{2x-5} - \frac{1}{x-1}\,dx$$

$$\lim_{R\to\infty}\int_5^R \frac{2}{2x-5} - \frac{1}{x-1}\,dx = \lim_{R\to\infty}\Big[\ln|2x-5| - \ln|x-1|\Big]_5^R$$

$$= \lim_{R\to\infty}\Big[\big(\ln|2(R)-5| - \ln|R-1|\big) - \big(\ln|2(5)-5| - \ln|5-1|\big)\Big]$$

$$= \lim_{R\to\infty}\left(\ln\left|\frac{2R-5}{R-1}\right| - \left(\ln\frac{2(5)-5}{5-1}\right)\right)$$

$$= \lim_{R\to\infty}\ln\frac{2R-5}{R-1} - \lim_{R\to\infty}\ln\frac{2(5)-5}{5-1} = \ln(2) - \ln\left(\frac{5}{4}\right)$$

(d) Determine whether the series $\sum_{n=5}^{\infty}\frac{3}{2n^2 - 7n + 5}$ converges or diverges. State the conditions of the test used for determining convergence or divergence.

Integral test

$$\int_5^\infty \frac{3}{2x^2-7x+5}\,dx = \ln(2) - \ln\left(\frac{5}{4}\right)$$

since $f(x)$ is continuous, decreasing, and positive on the interval $[5, \infty)$, the integral test can be used

$\int_5^\infty \frac{3}{2x^2-7x+5}\,dx$ converges to a value, $\sum_{n=5}^{\infty}\frac{3}{2n^2-7n+5}$ also converges by the integral test.

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NO CALCULATOR ALLOWED

5B1

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5. Let f be the function defined by $f(x) = \dfrac{3}{2x^2 - 7x + 5}$.

(a) Find the slope of the line tangent to the graph of f at $x = 3$.

$$f'(x) = \frac{-(3)(4x-7)}{(2x^2-7x+5)^2}$$

$$f'(3) = \frac{-3(12-7)}{(18-21+5)^2} = \frac{-3(5)}{2^2} = \boxed{-\frac{15}{4}}$$

(b) Find the x-coordinate of each critical point of f in the interval $1 < x < 2.5$. Classify each critical point as the location of a relative minimum, a relative maximum, or neither. Justify your answers.

$$f'(x) = \frac{-12x + 21}{(2x^2-7x+5)^2}$$

$-12x+21=0$

$x = \frac{21}{12} = \frac{7}{4}$

$(2x^2-7x+5)=0$

$(2x-5)(x-1)=0$

$x = \frac{5}{2}$ $x=1$

$x = \frac{7}{4}$ is a relative maximum because f' changes from positive to negative at that point

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Continue question 5 on page 19.

NO CALCULATOR ALLOWED

5B2

(c) Using the identity that $\frac{3}{2x^2 - 7x + 5} = \frac{2}{2x - 5} - \frac{1}{x - 1}$, evaluate $\int_5^{\infty} f(x)\,dx$ or show that the integral diverges.

$$\lim_{b\to\infty} \int_5^b \frac{2}{2x-5}\,dx - \int_5^b \frac{1}{x-1}\,dx$$

$$\lim_{b\to\infty} \Big[\ln|2x-5| - \ln|x-1|\Big]_5^b$$

$$\lim_{b\to\infty} \big(\ln|2b-5| - \ln|b-1|\big) - \ln 5 - \ln 4 \qquad \text{DNE}$$

diverges

(d) Determine whether the series $\sum_{n=5}^{\infty} \frac{3}{2n^2 - 7n + 5}$ converges or diverges. State the conditions of the test used for determining convergence or divergence.

$$\frac{3}{2n^2-7n+5} = \frac{\frac{3}{1}}{2n-5} + \frac{\frac{3}{5/2}}{n-1}$$

$$= \int \frac{3}{2n-5} + \frac{\frac{6}{5}}{n-1}$$

$$\frac{3}{2}\ln|2n-5| + \frac{6}{5}\ln|n-1|\Big|_5^b$$

GO ON TO THE NEXT PAGE.

5C1

5. Let f be the function defined by $f(x) = \dfrac{3}{2x^2 - 7x + 5}$. $8 - 14 + 5$

(a) Find the slope of the line tangent to the graph of f at $x = 3$.

$$f'(x) = \frac{-3(4x-7)}{(2x^2-7x+5)^2}$$

$$m_T\Big|_{x=3} = \frac{-3(12-7)}{(2\cdot 9 - 7\cdot 3+5)^2} = \frac{-15}{4}$$

(b) Find the x-coordinate of each critical point of f in the interval $1 < x < 2.5$. Classify each critical point as the location of a relative minimum, a relative maximum, or neither. Justify your answers.

$$-3(4x-7) = 0$$
$$-12x + 21 = 0$$
$$x = \frac{21}{12}$$

f': $+$ on $(1.1, \frac{21}{12})$, $-$ on $(\frac{21}{12}, 2)$

1.1 $\frac{21}{12}$ 2

relative maximum because f is increasing to the left and decreasing to the right.

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Continue question 5 on page 19.

5C2

(c) Using the identity that $\frac{3}{2x^2 - 7x + 5} = \frac{2}{2x - 5} - \frac{1}{x - 1}$, evaluate $\int_5^\infty f(x)\,dx$ or show that the integral diverges.

$$\int_5^\infty \left(\frac{2}{2x-5} - \frac{1}{x-1}\right)dx$$

(d) Determine whether the series $\sum_{n=5}^{\infty} \frac{3}{2n^2 \cancel{- 7n + 5}}$ converges or diverges. State the conditions of the test used for determining convergence or divergence.

$$= \sum_{n=6}^{\infty} \frac{3}{2n^2}$$

By test for Divergence $\sum_{n=5}^{\infty} \frac{3}{2n^2 - 7n + 5}$ diverges.

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Question 5

Overview

In this problem the function f is defined by $f(x) = \dfrac{3}{2x^2 - 7x + 5}$. In part (a) students were asked to find the slope of the line tangent to the graph of f at $x = 3$. Students needed to differentiate f and find the slope of the line tangent to the graph of f at $x = 3$ by evaluating $f'(3)$. [LO 2.3B/EK 2.3B1] In part (b) students were asked to find the x-coordinate of each critical point of f in the interval $1 < x < 2.5$ and to classify each critical point as the location of a relative minimum, a relative maximum, or neither, justifying these classifications. Students should have observed that f is differentiable on $1 < x < 2.5$ and found that $f'(x) = 0$ has just one solution, $x = \frac{7}{4}$, in this interval. Then students needed to determine that f has a relative maximum at $x = \frac{7}{4}$ by noting that f' changes sign from positive to negative at the critical point $x = \frac{7}{4}$. [LO 2.2A/EK 2.2A1] In part (c) students were given the partial fraction decomposition for $f(x)$ and asked to evaluate $\int_5^\infty f(x)\,dx$ or to show that the integral diverges. Students should have expressed the given improper integral as a limit of proper integrals, $\lim\limits_{b\to\infty}\int_5^b f(x)\,dx$, and used the partial fraction decomposition for $f(x)$ to find that $\int_5^b f(x)\,dx = \ln(2b-5) - \ln(b-1) - (\ln 5 - \ln 4) = \ln\left(\frac{2b-5}{b-1}\right) - \ln\left(\frac{5}{4}\right)$. Applying limit theorems, students needed to take the limit of this expression as $b \to \infty$ to find that the improper integral converges to $\ln\left(\frac{8}{5}\right)$. [LO 1.1C/EK 1.1C1-1.1C2, LO 3.2D/EK 3.2D1-3.2D2] In part (d) students were asked to determine whether the series $\sum\limits_{n=5}^{\infty} f(n)$ converges or diverges, stating the conditions of the test used for this determination. Students needed to combine the results of part (c) with the integral test or use a limit comparison test to the convergent p-series $\sum\limits_{n=5}^{\infty}\frac{1}{n^2}$ to find that the series $\sum\limits_{n=5}^{\infty} f(n)$ converges. For either test, students should have observed that the necessary conditions hold, namely that f is continuous, positive, and decreasing on $[5, \infty)$. [LO 4.1A/EK 4.1A6] This problem incorporates the following Mathematical Practices for AP Calculus (MPACs): reasoning with definitions and theorems, connecting concepts, implementing algebraic/computational processes, building notational fluency, and communicating.

Sample: 5A
Score: 9

The response earned all 9 points: 2 points in part (a), 2 points in part (b), 3 points in part (c), and 2 points in part (d). In part (a) the student earned both points by presenting a correct expression for $f'(x)$ and evaluating it correctly at $x = 3$, arriving at the value of $-\frac{15}{4}$. The student does not need to simplify the numerical expression $\dfrac{-3(4(3)-7)}{\left(2(3)^2 - 7(3) + 5\right)^2}$ to earn the points. The student chooses to simplify and does so correctly. In part (b) the student earned both points with the sentence at the bottom. In this sentence the x-coordinate is correctly identified,

Question 5 (continued)

the critical point is correctly classified as a relative maximum, and the student's justification is correct. In part (c) the student earned the antiderivative point with the correct expression presented on the right side of the second line. The limit expression point was earned with the expression on the left side of the second line. The student earned the answer point at the end of the last line. The expression $\ln(2) - \ln\left(\frac{5}{4}\right)$ does not need to be simplified. In part (d) the student earned both points for using the integral test to conclude that the given series converges and for stating the conditions for the integral test on the left side of the page.

Sample: 5B
Score: 6

The response earned 6 points: 2 points in part (a), 2 points in part (b), 2 points in part (c), and no points in part (d). In part (a) the student earned both points by presenting a correct expression for $f'(x)$ and evaluating it correctly at $x = 3,$ arriving at the value of $-\frac{15}{4}.$ The student does not need to simplify the numerical expression $\frac{-3(12-7)}{(18-21+5)^2}$ to earn the points. The student chooses to simplify and does so correctly. In part (b) the student earned both points with the boxed sentence in the lower left corner. In this sentence the x-coordinate is correctly identified, the critical point (which is identified in the work in the upper right corner) is correctly classified as a relative maximum, and the student's justification is correct. In part (c) the student earned the antiderivative point in the second line. The limit expression point was earned in the first line. The student does not determine the value of the limit correctly and did not earn the answer point. In part (d) the student does not draw any conclusion about the convergence of the given series, so no points were earned.

Sample: 5C
Score: 3

The response earned 3 points: 2 points in part (a), 1 point in part (b), no points in part (c), and no points in part (d). In part (a) the student earned both points by presenting a correct expression for $f'(x)$ and evaluating it correctly at $x = 3,$ arriving at the value of $-\frac{15}{4}.$ The student does not need to simplify the numerical expression $\frac{-3(12-7)}{(2\cdot 9 - 7\cdot 3 + 5)^2}$ to earn the points. The student chooses to simplify and does so correctly. In part (b) the student earned the first point for identifying $x = \frac{21}{12}$ as the x-coordinate of the critical point. The student does not provide sufficient justification for the classification of a relative maximum. The justification discusses the behavior of f as "increasing to the left and decreasing to the right" rather than focusing on the sign change of f' at $x = \frac{21}{12}.$ Thus, the second point was not earned. In part (c) the student does not evaluate the integral, so no points were earned. In part (d) the student makes an incorrect claim about the convergence of the series without any support for the conclusion. No test is used, and no conditions are stated. The student did not earn any points.

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Sample Student Responses and Scoring Commentary

Inside:

☑ Free Response Question 6

☑ Scoring Guideline

☑ Student Samples

☑ Scoring Commentary

Question 6

(a) $f(0) = 0$

$f'(0) = 1$

$f''(0) = -1(1) = -1$

$f'''(0) = -2(-1) = 2$

$f^{(4)}(0) = -3(2) = -6$

The first four nonzero terms are

$$0 + 1x + \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 + \frac{-6}{4!}x^4 = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}.$$

The general term is $\dfrac{(-1)^{n+1}x^n}{n}$.

$$3 : \begin{cases} 1 : f''(0),\ f'''(0),\ \text{and } f^{(4)}(0) \\ 1 : \text{verify terms} \\ 1 : \text{general term} \end{cases}$$

(b) For $x = 1$, the Maclaurin series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

The series does not converge absolutely because the harmonic series diverges.

The series alternates with terms that decrease in magnitude to 0, and therefore the series converges conditionally.

2 : converges conditionally with reason

(c) $$\int_0^x f(t)\,dt = \int_0^x \left(t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots + \frac{(-1)^{n+1}t^n}{n} + \cdots \right) dt$$

$$= \left[\frac{t^2}{2} - \frac{t^3}{3 \cdot 2} + \frac{t^4}{4 \cdot 3} - \frac{t^5}{5 \cdot 4} + \cdots + \frac{(-1)^{n+1}t^{n+1}}{(n+1)n} + \cdots \right]_{t=0}^{t=x}$$

$$= \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{20} + \cdots + \frac{(-1)^{n+1}x^{n+1}}{(n+1)n} + \cdots$$

$$3 : \begin{cases} 1 : \text{two terms} \\ 1 : \text{remaining terms} \\ 1 : \text{general term} \end{cases}$$

(d) The terms alternate in sign and decrease in magnitude to 0. By the alternating series error bound, the error $\left| P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right) \right|$ is bounded by the magnitude of the first unused term, $\left| -\dfrac{(1/2)^5}{20} \right|$.

Thus, $\left| P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right) \right| \le \left| -\dfrac{(1/2)^5}{20} \right| = \dfrac{1}{32 \cdot 20} < \dfrac{1}{500}$.

1 : error bound

6A1

NO CALCULATOR ALLOWED

6A1

$$f(0) = 0$$
$$f'(0) = 1$$
$$f^{(n+1)}(0) = -n \cdot f^{(n)}(0) \text{ for all } n \geq 1$$

6. A function f has derivatives of all orders for $-1 < x < 1$. The derivatives of f satisfy the conditions above. The Maclaurin series for f converges to $f(x)$ for $|x| < 1$.

(a) Show that the first four nonzero terms of the Maclaurin series for f are $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$, and write the general term of the Maclaurin series for f.

The Maclaurin series $P(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots + \frac{f^{(n)}}{n!}x^n + \cdots$

$f(0)=0$, $f'(0)=1$, $f''(0) = -1$, $f'''(0) = 2$, $f^{(4)}(0) = -6$

$P_4(x) = 0 + \frac{1}{1}x + \frac{-1}{2}x^2 + \frac{2}{3!}x^3 + \frac{-6}{4!}x^4 = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$

The nth term is: $\frac{f^{(n)}(0)}{n!}x^n = \frac{-(n-1)f^{(n-1)}(0)}{n!}x^n = \frac{(-1)^{n-1}x^n}{n}$

this is the general term: $\frac{(-1)^{n-1}x^n}{n}$

(b) Determine whether the Maclaurin series described in part (a) converges absolutely, converges conditionally, or diverges at $x = 1$. Explain your reasoning.

at $x=1$, the series is $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$;

① Because $\lim_{n\to\infty} \left|\frac{(-1)^{n-1}}{n}\right| = 0$, $\left|\frac{1}{n+1}\right| < \left|\frac{1}{n}\right|$ for all $n > 0$, by the alternating series convergence theorem, it converges.

② However, $\sum_{n=1}^{\infty}\left|\frac{(-1)^{n-1}}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n}$ is the known diverging harmonic series;

So the original Maclaurin series converges conditionally

Continue question 6 on page 21.

6A₂ NO CALCULATOR ALLOWED 6A₂

(c) Write the first four nonzero terms and the general term of the Maclaurin series for $g(x) = \int_0^x f(t)\, dt$.

$g(x) = \int_0^x f(t)\,dt = \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{12}x^4 - \frac{1}{20}x^5 + \cdots + \frac{(-1)^n}{n(n-1)}x^n + \cdots$

$(\,g(0)=0\,)$

(d) Let $P_n\left(\frac{1}{2}\right)$ represent the nth-degree Taylor polynomial for g about $x = 0$ evaluated at $x = \frac{1}{2}$, where g is the function defined in part (c). Use the alternating series error bound to show that

$\left|P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right)\right| < \frac{1}{500}.$

By the alternating series error bound, the error $\epsilon_4(x)$, satisfies: $\epsilon_4(x) = |P_4(x) - g(x)| < b_5(x)$

where $b_5(x)$ is $\left|-\frac{1}{20}x^5\right|$, the absolute value of the 5th degree term in the Taylor polynomial.

at $x = \frac{1}{2}$, $b_5(\frac{1}{2}) = \frac{1}{20} \times \frac{1}{32} = \frac{1}{640} < \frac{1}{500}$

So $|P_4(\frac{1}{2}) - g(\frac{1}{2})| = \epsilon_4(x) < \frac{1}{640} < \frac{1}{500}$.

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$$f(0) = 0$$
$$f'(0) = 1$$
$$f^{(n+1)}(0) = -n \cdot f^{(n)}(0) \text{ for all } n \geq 1$$

6. A function f has derivatives of all orders for $-1 < x < 1$. The derivatives of f satisfy the conditions above. The Maclaurin series for f converges to $f(x)$ for $|x| < 1$.

(a) Show that the first four nonzero terms of the Maclaurin series for f are $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$, and write the general term of the Maclaurin series for f.

general term

$\sum_{n=1}^{\infty} (-1)^{N+1} \frac{x^N}{N}$

Since $f(0)=0$

$f'(0)=1$

$f^{(2)}(0) = -1 \cdot f' = 1$

$f^{(3)}(0) =$

(b) Determine whether the Maclaurin series described in part (a) converges absolutely, converges conditionally, or diverges at $x = 1$. Explain your reasoning.

at $x=1$

$\sum_{n=1}^{\infty} (-1)^{N+1} \frac{(1)^N}{N} \Rightarrow \frac{(-1)^{N+1}}{N}$ = which is an alternating harmonic Series So it converges conditionally

Absolute Must have

$\sum_{n=1}^{\infty} \frac{1}{N}$ coverge but $\frac{1}{N}$ is harmonic as $p \leq 1$ so diverges

$\frac{(-1)^{N+1}}{n+1} < \frac{(-1)^{N+1}}{N}$

$\lim_{n \to \infty} \frac{(-1)^{N+1}}{N} = 0$

Continue question 6 on page 21.

6B₂ NO CALCULATOR ALLOWED 6B₂

(c) Write the first four nonzero terms and the general term of the Maclaurin series for $g(x) = \int_0^x f(t)\,dt$.

$$\sum_{n=1}^{\infty} (-1)^{N+1} \frac{x^{N+1}}{n \cdot (n+1)}$$

Non zero term for

$$g(x) = \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{20}$$

(d) Let $P_n\left(\frac{1}{2}\right)$ represent the nth-degree Taylor polynomial for g about $x = 0$ evaluated at $x = \frac{1}{2}$, where g is the function defined in part (c). Use the alternating series error bound to show that

$\left|P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right)\right| < \frac{1}{500}$.

$$\left| P_4\left(\tfrac{1}{2}-\right) g\left(\tfrac{1}{2}\right)\right| < \frac{1}{500}$$

$$P_4(x) = \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12}$$

$$P\left(\tfrac{1}{2}\right) = \frac{\frac{1}{4}}{2} - \frac{\frac{1}{8}}{6} + \frac{\frac{1}{16}}{12}$$

$$f(0) = 0$$
$$f'(0) = 1$$
$$f^{(n+1)}(0) = -n \cdot f^{(n)}(0) \text{ for all } n \geq 1$$

6. A function f has derivatives of all orders for $-1 < x < 1$. The derivatives of f satisfy the conditions above. The Maclaurin series for f converges to $f(x)$ for $|x| < 1$.

(a) Show that the first four nonzero terms of the Maclaurin series for f are $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$, and write the general term of the Maclaurin series for f.

MacLaurin Series: $\frac{f^{(n)}(0) \cdot x^n}{n!}$

$n=1$ $\frac{f_1'(0) \cdot x^1}{1!}$ $n=2$ $\frac{-1 \cdot 1x^2}{2!}$ $n=3$ $\frac{-2 \cdot -1x^3}{3 \cdot 2}$ $n=4$ $\frac{-3 \cdot 2x^4}{4 \cdot 3 \cdot 2}$

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$$

(b) Determine whether the Maclaurin series described in part (a) converges absolutely, converges conditionally, or diverges at $x = 1$. Explain your reasoning.

$$\frac{-n f^{(n)}(0)\, x^{n+1}}{(n+1)!} \qquad \frac{-n f^{(n)}(0)}{(n+1)!}$$

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$ This series converges conditionally because when $x=1$ it can be compared to alternating harmonic, which converges conditionally

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6C2

NO CALCULATOR ALLOWED

6C2

(c) Write the first four nonzero terms and the general term of the Maclaurin series for $g(x) = \int_0^x f(t)\,dt$.

$$\frac{f^{(n)}(0)\,x^n}{n!}$$

$\cancel{n=1}$ $n=2$ $n=3$ $n=4$ $n=5$

$\cancel{0}$ $\frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \frac{x^5}{5}$

(d) Let $P_n\left(\frac{1}{2}\right)$ represent the nth-degree Taylor polynomial for g about $x = 0$ evaluated at $x = \frac{1}{2}$, where g is the function defined in part (c). Use the alternating series error bound to show that

$\left|P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right)\right| < \frac{1}{500}$.

$$P_4\left(\tfrac{1}{2}\right) = \frac{\left(\frac{1}{2}\right)^5}{5} = \frac{\frac{1}{32}}{5} = \frac{1}{32\cdot 5}$$

$$\begin{array}{r} 32 \\ \times\ 5 \\ \hline 160 \end{array}$$

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Question 6

Overview

In this problem students were presented with a function f that has derivatives of all orders for $-1 < x < 1$ such that $f(0) = 0,$ $f'(0) = 1,$ and $f^{(n+1)}(0) = -n \cdot f^{(n)}(0)$ for all $n \geq 1.$ It is also stated that the Maclaurin series for f converges to $f(x)$ for $|x| < 1.$ In part (a) students were asked to verify that the first four nonzero terms of the Maclaurin series for f are $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$ and to write the general term of this Maclaurin series. The nth-degree term of the Taylor polynomial for f about $x = 0$ is $\frac{f^{(n)}(0)}{n!}x^n.$ $f(0) = 0$ and $f'(0) = 1$ are given, and the given recurrence relation for $f^{(n+1)}(0)$ can be readily applied to see that $f''(0) = -1,$ $f'''(0) = 2,$ $f^{(4)}(0) = -6,$ and $f^{(n)}(0) = (-1)^{n+1}(n-1)!$. Using these derivative values, students needed to confirm that the first four nonzero terms of the Maclaurin series for f are as given, and that the general term is $\frac{(-1)^{n+1}x^n}{n}.$ [LO 4.2A/EK 4.2A1] In part (b) students were asked to determine, with explanation, whether the Maclaurin series for f converges absolutely, converges conditionally, or diverges at $x = 1.$ Substituting $x = 1,$ students should have obtained that the Maclaurin series for f evaluated at $x = 1$ is $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}.$ Students needed to conclude that this series converges conditionally, noting that the series converges by the alternating series test and that $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right|$ is the divergent harmonic series. [LO 4.1A/EK 4.1A4-4.1A6] In part (c) students were asked to find the first four nonzero terms and the general term of the Maclaurin series for $g(x) = \int_0^x f(t)\,dt.$ Students needed to find these terms by integrating the Maclaurin series for f term-by-term. [LO 4.2B/EK 4.2B5] In part (d) using the function g defined in part (c), the expression $P_n\left(\frac{1}{2}\right)$ represents the nth-degree Taylor polynomial for g about $x = 0$ evaluated at $x = \frac{1}{2}.$ Students were directed to use the alternating series error bound to show that $\left| P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right) \right| < \frac{1}{500}.$ Students may have observed that the terms of the Taylor polynomial for g about $x = 0,$ evaluated at $x = \frac{1}{2},$ alternate in sign and decrease in magnitude to 0. Thus, the alternating series error bound can be applied to see that $\left| P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right) \right| \leq \left| -\frac{(1/2)^5}{20} \right| = \frac{1}{32 \cdot 20},$ showing that the error in the approximation is less than $\frac{1}{500}.$ [LO 4.1B/EK 4.1B2] This problem incorporates the following Mathematical Practices for AP Calculus (MPACs): reasoning with definitions and theorems, connecting concepts, implementing algebraic/computational processes, building notational fluency, and communicating.

Question 6 (continued)

Sample: 6A
Score: 9

The response earned all 9 points: 3 points in part (a), 2 points in part (b), 3 points in part (c), and 1 point in part (d). In part (a) the student computes the numerical values of the derivatives in line 3 and earned the first point. The student uses the derivatives in line 4 to verify the given expression and earned the second point. The student produces a correct general term in line 5 using -1 with an exponent of $n-1$ rather than $n+1$, which is still correct. The student earned the third point. In part (b) the student correctly uses the alternating series test to draw a conclusion of converges, identifies the harmonic series as divergent, and draws the correct conclusion of converges conditionally. The student earned both points. In part (c) the student produces the first four terms and earned the first 2 points. The student presents a correct general term using indices of n rather than $n+1$ and earned the third point. In part (d) the student correctly computes the alternating series error and identifies it as the bound on the error.

Sample: 6B
Score: 6

The response earned 6 points: 1 point in part (a), 2 points in part (b), 3 points in part (c), and no point in part (d). In part (a) the student does not present the numerical values of the derivatives or include a proper verification. The student did not earn the first 2 points. The student produces a correct general term and earned the third point. The student is not penalized for using both lower and upper cases for n in this question. In part (b) the student correctly identifies the series as the alternating harmonic and the absolute value series as the harmonic. The student draws the correct conclusion of converges conditionally. The student earned both points. In part (c) the student produces the first four terms and the general term. All 3 points were earned. In part (d) the student does not produce the numerical error value to verify that the error is less than $\frac{1}{500}$. The student did not earn the point.

Sample: 6C
Score: 3

The response earned 3 points: 2 points in part (a), 1 point in part (b), no points in part (c), and no point in part (d). In part (a) the student computes the numerical values of the derivatives embedded in line 2 and earned the first point. The student's use of derivatives in the verification line is correct and earned the second point. The student does not present a general term, so the third point was not earned. In part (b) the student concludes that the alternating series converges conditionally, but the student does not explicitly address the series of absolute values of the terms. The student earned 1 of the 2 points. In part (c) the student produces four terms, but only the first term is correct. The student did not earn either of the first 2 points. The student does not produce a sufficient general term because line 1 is formulaic. The student did not earn the third point. In part (d) the student correctly computes the error bound based on the work in part (c) but does not connect $P_4\left(\frac{1}{2}\right) = \frac{1}{32 \cdot 5}$ to the error. The student did not earn the point.